EGMO 2025
European Girls' Mathematical Olympiad
KOSOVA

Language: **English**

Day: **1**

Sunday, April 13, 2025

Problem 1. For a positive integer N, let $c_1 < c_2 < \cdots < c_m$ be all the positive integers smaller than N that are coprime to N. Find all $N \geq 3$ such that

$$\gcd(N, c_i + c_{i+1}) \neq 1$$

for all $1 \leq i \leq m - 1$.

Here $\gcd(a, b)$ is the largest positive integer that divides both a and b. Integers a and b are coprime if $\gcd(a, b) = 1$.

Problem 2. An infinite increasing sequence $a_1 < a_2 < a_3 < \cdots$ of positive integers is called *central* if for every positive integer n, the arithmetic mean of the first a_n terms of the sequence is equal to a_n.

Show that there exists an infinite sequence b_1, b_2, b_3, ... of positive integers such that for every central sequence a_1, a_2, a_3, ..., there are infinitely many positive integers n with $a_n = b_n$.

Problem 3. Let ABC be an acute triangle. Points B, D, E, and C lie on a line in this order and satisfy $BD = DE = EC$. Let M and N be the midpoints of AD and AE, respectively. Suppose triangle ADE is acute, and let H be its orthocentre. Points P and Q lie on lines BM and CN, respectively, such that D, H, M, and P are concyclic and pairwise different, and E, H, N, and Q are concyclic and pairwise different. Prove that P, Q, N, and M are concyclic.

The orthocentre of a triangle is the point of intersection of its altitudes.

Time: 4 hours and 30 minutes
Each problem is worth 7 points